# PHYS-100C, Lecture 5

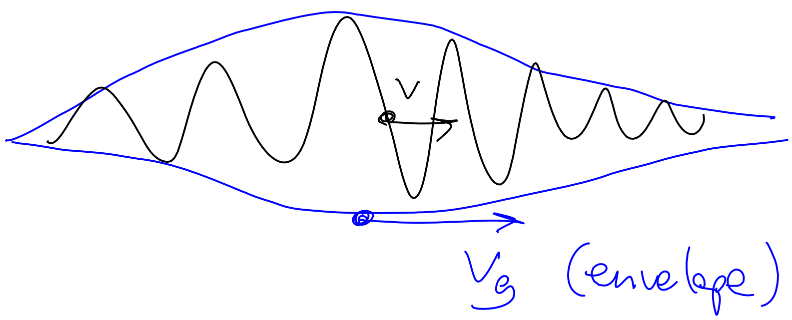
Tuesday, April 14, 2009
6:30 AM

## * Dispersion:

$\varepsilon$ ($\Rightarrow n, v$) depends on $\omega$

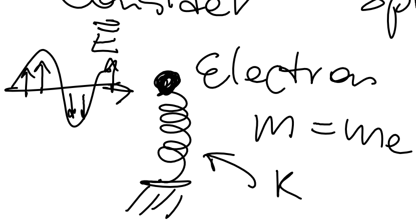
Phase velocity $v = \dfrac{\omega}{k}$

Group velocity $v_g = \dfrac{\partial \omega}{\partial k}$




Why does $\varepsilon$ (and $v$, $n$ etc.) depend on $\omega$?

Consider "spring" model of electron:




$\omega = \sqrt{\dfrac{k}{m_e}}$

Eq. of motion:

$$\underbrace{m\frac{\partial^2 x}{\partial t^2}}_{F=ma} + \underbrace{m\gamma\frac{\partial x}{\partial t}}_{\text{dissipation}} + \underbrace{m\omega_0^2 x}_{kx} = \underbrace{eE_0 \cdot \cos\omega t}_{\text{driving force}}$$

General Solution: $x = \tilde{x}_0 \cdot e^{-i\omega t}$

$$-\omega^2 x - i\gamma x + \omega_0^2 x = \frac{eE_0}{m} e^{-i\omega t}$$

$$\tilde{x}_0 = \frac{e/m}{\omega_0^2 - \omega^2 - i\gamma} E_0$$

Polarization $\tilde{p} = e\tilde{x}$

$$\tilde{P} = \sum_{\substack{\text{ALL} \\ \text{ELECTRONS/VOLUME}}} \tilde{p} = \varepsilon_0 \tilde{\chi}_e \tilde{E}$$

$$\tilde{\varepsilon} = \varepsilon_0 (1 + \tilde{\chi}_e)$$

$$\tilde{\varepsilon}_R = 1 + \frac{Ne^2}{m\varepsilon_0} \cdot \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma\omega}$$

Planar wave $E = E_0 \cdot e^{i(\tilde{k}z - \omega t)}$

where $\tilde{k} = \sqrt{\tilde{\varepsilon}\mu_0}\,\omega = k + i\frac{\alpha}{2}$

$$E = E_0 \cdot e^{-\frac{\alpha z}{2}} \cdot e^{i(kz - \omega t)}$$

$\alpha$ is absorption coefficient

$$n = \frac{ck}{\omega}$$

for small dispersive corrections:

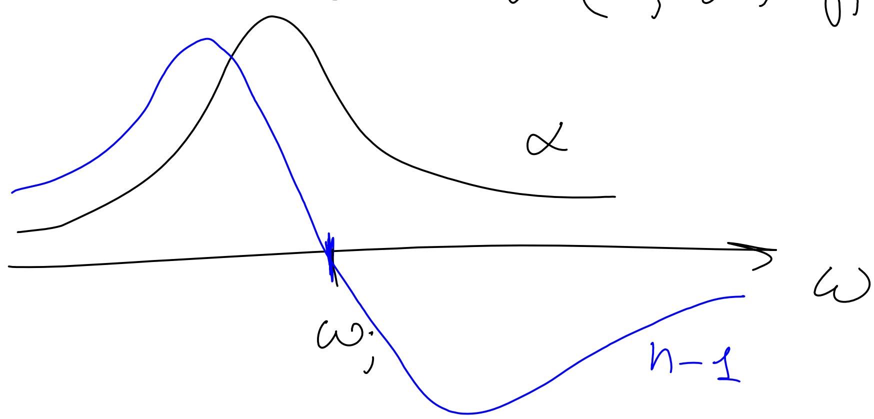
$$\sqrt{1+\varepsilon} \simeq 1 + \varepsilon/2$$

$$\tilde{k} = \frac{\omega}{c}\left[1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

$$n = \frac{ck}{\omega} \cong 1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2}$$

and

$$\alpha \cong \frac{Ne^2\omega^2}{m\varepsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}$$




Far away from resonances,

$$n \sim 1 + \dots \sum \frac{\dots}{\omega_j^2 - \omega^2}$$

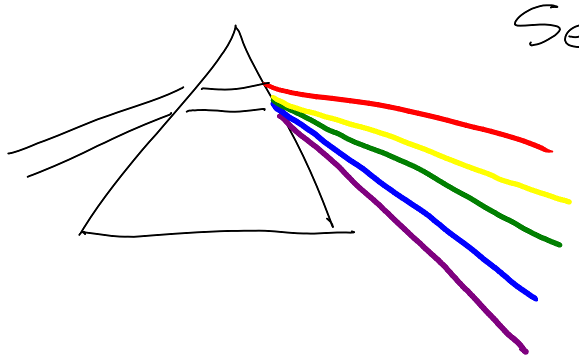
$\omega_j$ are in UV range

for visible light $\omega < \omega_j$

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2(1 - \omega^2/\omega_j^2)} \approx \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

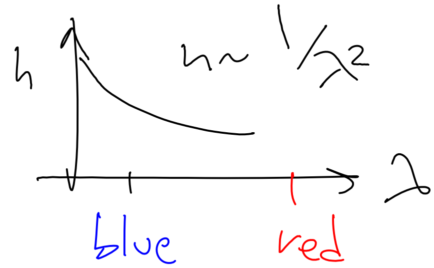
$$n = 1 + A\left(1 + B/\lambda^2\right)$$ Cauchy's FORMULA

* Physics consequences:



See Cover ART "Dark side of the Moon" by PINK FLOYD

n increases for shorter wavelengths (blue/violet)




RAINBOWS:

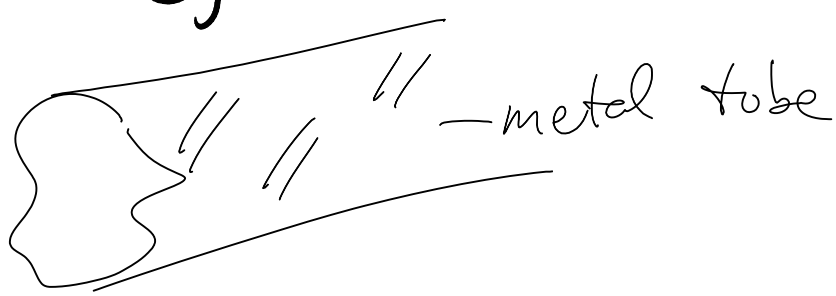
UV light

(bad for skin, can damage DNA)
gets substantially absorbed by atmosphere (since $\omega_i \approx$ UV)

For very large frequencies

$\omega > \omega_i \qquad n < 1 \qquad$ (X-RAYS)

* Waveguides



Perfect Conductor:

$E=0$, $B=0$ inside

Boundary conditions:

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma \qquad \text{(i)}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \text{(ii)}$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad \text{(iii)}$$

$$\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = K_f \times n \qquad \text{(iv)}$$

Or $E^{\parallel}=0$, $B^{\perp}=0$ (ii), (iii)

(since $E=0$, $B=0$ inside)

In interior of the waveguide

$$\bar{\nabla}\cdot\bar{E}=0 \qquad \bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$$

$$\bar{\nabla}\cdot\bar{B}=0 \qquad \nabla\times B = \frac{1}{c^2}\frac{\partial E}{\partial t}$$

No free charges, currents, assume vacuum (thus $1/c^2$)

$$\tilde{E} = \tilde{E}_0 \cdot e^{i(kz-\omega t)}$$

$$\tilde{B} = \tilde{B}_0 \cdot e^{i(kz-\omega t)}$$

$E$, $B$ are not generally transverse (which was a result of plane waves with no x-y dependence for $E$, $B$)

$$\nabla\times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

For (iii):

$$(z):\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \qquad (1)$$

$$(x):\ \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x \qquad (2)$$

$$(y):\ ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y \qquad (3)$$

$$\left(\text{Since } \frac{\partial E_y}{\partial z} = ikE_y \quad \frac{\partial E_x}{\partial z} = ikE_x\right)$$

Similarly for Eq.(iv):

$$(z):\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \qquad (4)$$

$$(x):\ \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x \qquad (5)$$

$$(y):\ ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \qquad (6)$$

From (3) & (5), eliminate $B_y$:

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

From (2) & (6), eliminate $B_x$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

Similarly for $B_x, B_y$:

(2) and (6):

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\cdot\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

$E_z$ & $B_z$ determine everything else.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right) E_z, B_z = 0$$

$E_z = 0 \Rightarrow$ TE waves

$B_z = 0 \Rightarrow$ TM waves

$E_z = B_z = 0 \Rightarrow$ TEM waves (cannot occur in hollow WG)

*** Rectangular Waveguide:**